THE TOPOLOGY OF TORIC ORIGAMI MANIFOLDS

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ABSTRACT. A folded symplectic form on a manifold is a closed 2-form with the mildest possible degeneracy along a hypersurface. A special class of folded symplectic manifolds are the origami symplectic manifolds, studied by Cannas da Silva, Guillemin and Pires, who classified toric origami manifolds by combinatorial origami templates. In this paper, we examine the topology of toric origami manifolds that have acyclic origami template and coörientable folding hypersurface. We prove that the cohomology is concentrated in even degrees, and that the equivariant cohomology satisfies the GKM description. Finally we show that toric origami manifolds with coörientable folding hypersurface provide a class of examples of Masuda and Panov's torus manifolds.

CONTENTS

1. INTRODUCTION

Toric symplectic manifolds are a useful class of examples for testing general theories and making explicit computations. Statements and proofs of important theorems often simplify in the case of toric manifolds. Delzant's classification of toric symplectic manifolds in terms of convex polytopes allows the translation of geometric and topological questions into combinatorial ones. In this paper, we study toric actions in the category of **folded symplectic manifolds**. Relaxing the requirement that the manifold be symplectic broadens the class of manifolds with toric actions. The mildest degeneracy is to allow the 2-form to be zero along a hypersurface. In this instance, there remains enough geometric structure to be able to classify such **toric origami manifolds** combinatorially.

In this paper, we study the topology of a particular class of toric origami manifolds, those with **acyclic template** and **coörientable fold**. For such manifolds, we prove that the ordinary cohomology is concentrated in even degrees (Theorem 3.6). This allows us to deduce a variety of facts about the **equivariant** cohomology of these manifolds, and in particular to describe the equivariant cohomology ring combinatorially (Theorem 4.14). Our class of toric origami manifolds does fit into the framework of **torus manifolds** (Theorem 6.2). The origami structure allows us to give explicit inductive proofs. We plan to use similar geometric techniques to study the non-coörientable

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and non-acyclic cases. We hope that this approach will also generalize to a class of torus manifolds that arise from combinatorial origami templates, in the same way that some torus manifolds arise from combinatorial polytopes.

The remainder of this paper is organized as follows. In Section 2, we review the symplectic and folded symplectic geometry underlying our work. We then provide a framework for computing the ordinary and equivariant cohomology of origami manifolds with coörientable folding hypersurface and acyclic template in Sections 3 and 4. After describing these computations explicitly in Section 5, we describe the relationship of our work with the toric topology literature in Section 6.

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2. ORIGAMI MANIFOLDS

2.1. **Symplectic manifolds.** We begin with a very quick review of symplectic geometry. Let M be a **manifold** equipped with a **symplectic form** $\omega \in \Omega^2(M)$ that is closed ($d\omega = 0$) and non-degenerate. In particular, the non-degeneracy condition implies that M must be an even-dimensional manifold. The simplest examples include

(1) $M = \mathbb{S}^2 = \mathbb{CP}^1$ with $\omega_p(\mathcal{X}, \mathcal{Y}) =$ signed area of the parallelogram spanned by $\mathcal{X}$ and $\mathcal{Y}$;
(2) M any compact orientable surface with ω the area form; and
(3) $M = \mathbb{R}^{2d}$ with $\omega = \sum dx_i \wedge dy_i$. The Darboux Theorem says that every symplectic manifold has local coördinates so that ω is of this standard form.

Suppose that a compact connected abelian group $\mathbb{T} = (\mathbb{S}^1)^n$ acts on M preserving ω. The action is **weakly Hamiltonian** if for every vector $\xi \in \mathfrak{t}$ in the Lie algebra $\mathfrak{t}$ of T, the vector field

$$\mathcal{X}_\xi(p) = \frac{d}{dt}\Big[\exp(t\xi)\cdot p\Big]\Big|_{t=0}$$

is a **Hamiltonian vector field**. That is, we require

$$\omega(\mathcal{X}_\xi, \cdot) = d\phi^\xi \tag{2.1}$$

to be an exact one-form[1]. Thus each ϕ^ξ is a smooth function on M defined by the differential equation (2.1), so determined up to a constant. Taking them together, we may define a **moment map**

$$\begin{array}{rcc} \Phi : M & \longrightarrow & \mathfrak{t}^* \\ p & \mapsto & \left(\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathbb{R} \\ \xi & \mapsto & \phi^\xi(p) \end{array} \right). \end{array}$$

The action is **Hamiltonian** if the moment map Φ can be chosen to be a $\mathbb{T}$-invariant map. Atiyah and Guillemin-Sternberg have shown that when M is a compact hamiltonian $\mathbb{T}$-manifold, the image $\Phi(M)$ is a convex polytope, and is the convex hull of the images of the fixed points $\Phi(M^{\mathbb{T}})$ [A, GS].

For an **effective**[2] Hamiltonian $\mathbb{T}$ action on M, $\dim(\mathbb{T}) \leq \frac{1}{2}\dim(M)$. We say that the action is **toric** if this inequality is in fact an equality. A symplectic manifold M with a toric Hamiltonian $\mathbb{T}$ action is called a **symplectic toric manifold**. Delzant used the moment polytope to classify symplectic toric manifolds.

A polytope Δ in $\mathbb{R}^n$ is **simple** if there are n **edges** adjacent to each **vertex**, and it is **rational** if the edges have rational slope. A simple polytope is **smooth at a vertex** if the n primitive vectors

[1] The one-form $\omega(\mathcal{X}_\xi, \cdot)$ is automatically closed because the action preserves ω.
[2] An action is effective if no non-trivial subgroup acts trivially.

parallel to the edges at the vertex span the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$ over $\mathbb{Z}$. It is **smooth** if it is smooth at each vertex. A simple rational smooth convex polytope is called a **Delzant polytope**. We may now state Delzant's result.

Theorem 2.2 (Delzant [De]). *There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{compact toric} \\ \textit{symplectic manifolds} \end{array} \right\} \rightsquigarrow \left\{ \textit{ Delzant polytopes } \right\},$$

up to equivariant symplectomorphism on the left-hand side and affine equivalence on the right-hand side.

2.2. **Origami manifolds.** We now relax the non-degeneracy condition on ω, following [CGP]. A **folded symplectic form** on a $2n$-dimensional manifold M is a 2-form $\omega \in \Omega^2(M)$ that is closed ($d\omega = 0$), whose top power ω^n vanishes transversely on a subset Z and whose restriction to points in Z has maximal rank. The transversality forces Z to be a codimension 1 embedded submanifold of M. We call Z the **folding hypersurface** or **fold**.

The simplest examples of folded symplectic manifolds include the following.

1. Euclidean space $M = \mathbb{R}^{2d}$ has folded symplectic form $\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{d} dx_i \wedge dy_i$. The Folded Darboux Theorem says that at points in $Z = \{x_1 = 0\}$, every folded symplectic manifold has local coördinates so that ω is of this standard form.
2. Any even-dimensional sphere $M = \mathbb{S}^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ may be equipped with the form $\omega_{\mathbb{C}^n} \oplus 0$. The folding hypersurface is the equator $Z = \mathbb{S}^{2n-1} \subset \mathbb{C}^n \oplus \{0\}$.
3. Any compact surface M can be equipped with a folded symplectic form with Z a union of circles. This includes non-orientable surfaces. For example, $\mathbb{R}P^2$ can be equipped with a folded symplectic form so that Z is a single circle.

Let $i : Z \hookrightarrow M$ be the inclusion of Z as a submanifold of M. Our assumptions imply that $i^*\omega$ has a 1-dimensional kernel on Z. This line field is called the **null foliation** on Z. An **origami manifold** is a folded symplectic manifold (M, ω) whose null foliation is fibrating: $Z \xrightarrow{\pi} B$ is a fiber bundle with orientable circle fibers over a compact base B. The form ω is called an **origami form** and the bundle π is called the **null fibration**. In the examples above, the first is not origami because the fibers are $\mathbb{R}$ rather than $\mathbb{S}^1$, but the second and third are origami. In the second example, the null fibration is the Hopf bundle $\mathbb{S}^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$, and in the third example, the base B consists of isolated points.

The definition of a Hamiltonian action only depends on ω being closed. Thus, in the folded framerwork, we may define moment maps and toric actions exactly as in Section 2.1. For example, the action $\mathbb{T}^2 \circlearrowright \mathbb{S}^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$ given by rotation on the $\mathbb{C}^2$ coördinates is Hamiltonian with moment map

$$\Phi(z_1, z_1, t) = \left(|z_1|^2, |z_2|^2\right).$$

The image of this map is shown in Figure 2.3 below.

An oriented origami manifold M with fold Z may be **unfolded** into a symplectic manifold as follows. Consider the closures of the connected components of $M \setminus Z$, a manifold with boundary which consists of two copies of Z. We collapse the fibers of the null fibration by identifying the boundary points that are in the same fiber of the null fibration of each individual copy of Z. The result, $M_0 := (M \setminus Z) \cup B_1 \cup B_2$, is a (disconnected) smooth manifold that can be naturally endowed with a symplectic form which on $M_0 \setminus (B_1 \cup B_2)$ coincides with the origami form on $M \setminus Z$. Because this can be achieved using symplectic cutting techniques, the resulting manifold M_0 is called the **symplectic cut space** (and its connected components the **symplectic cut pieces**), and the process is also called **cutting**. An example of cutting a 2-torus is shown in Figure 2.4. The symplectic cut space of a nonorientable origami manifold is the $\mathbb{Z}_2$-quotient of the symplectic cut space of its orientable double cover.

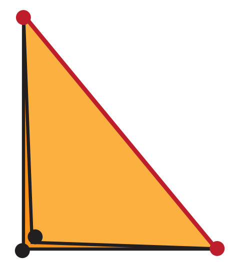

FIGURE 2.3. The moment map image for the $\mathbb{T}^2$ action on $\mathbb{S}^4$. The image consists of two overlapping copies of a triangle, which we have slightly unfolded. The red hypotenuse is the image of the equator $\mathbb{S}^3$. Every other point in the image has two connected components mapping to it, one from the northern hemisphere and the other from the southern.

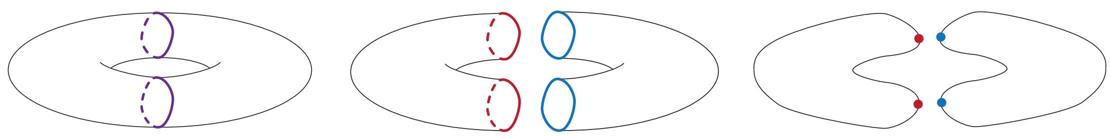

FIGURE 2.4. The torus, with fold $Z = \mathbb{S}^1 \cup \mathbb{S}^1$ in purple; the middle step before collapsing, the two copies of Z are in blue and purple; and the final cut space $M_0 = \mathbb{S}^2 \cup \mathbb{S}^2$ with B_1 in red and B_2 in blue.

In the example shown in Figure 2.3, unfolding the origami $\mathbb{S}^4$ yields $\mathbb{CP}^2 \sqcup \overline{\mathbb{CP}^2}$. This is suggested by the image of the moment map: the moment image of each toric $\mathbb{CP}^2$ (regardless of orientation) is a triangle. The cut space M_0 of an oriented origami manifold (M, ω) inherits a natural orientation. It is the orientation on M_0 induced from the orientation on M that matches the symplectic orientation on the symplectic cut pieces corresponding to the subset of $M \setminus Z$ where $\omega^n > 0$ and the opposite orientation on those pieces where $\omega^n < 0$.

We now restrict to origami manifolds whose fold is **coörientable**: that is, the fold has an orientable neighborhood. In this case, there is a choice of a coördinate t and a small enough collar neighborhood of $Z \subset M$ of the form $Z \times (-\varepsilon, \varepsilon)$ such that in these coördinates the origami form can be written involving only t^2 and not t. As a result, the involution $t \mapsto -t$ is a symplectomorphism. This then induces a symplectic involution γ on a tubular neighborhood $\mathcal{U}$ of $B \subset M_0$, preserving B; we call this a **model involution** of $\mathcal{U}$.[3]

Proposition 2.5 ([CGP, Props. 2.5 & 2.7]). *Let M be a (possibly disconnected) symplectic manifold with a codimension two symplectic submanifold*[4] *B and a model involution γ of a tubular neighborhood $\mathcal{U}$ of B. Then there is an origami manifold $\widetilde{M}$ such that M is the symplectic cut space of $\widetilde{M}$. Moreover, this manifold is unique up to origami–symplectomorphism.*

This newly-created fold $Z \subset \widetilde{M}$ involves the radial projectivized normal bundle of $B \subset M$, so we call the origami manifold $\widetilde{M}$ the **radial blow-up** of M through (γ, B). The cutting operation and the radial blow-up operation are in the following sense inverse to each other.

[3] For noncoörientable folds, a model involution must satisfy additional conditions.

[4] In the coörientable case, we have $B = B_1 \cup B_2$ and the model involution γ maps a tubular neighborhood of B_1 to one of B_2 and vice versa.

Proposition 2.6 ([CGP, Prop. 2.37]). *Let* M *be an origami manifold with cut space* M_0. *The radial blow-up* $\widetilde{M_0}$ *is origami-symplectomorphic to* M.

There exist Hamiltonian versions of these two operations which may be used to see that the moment map Φ for an origami manifold M coincides, on each connected component of $M \setminus Z$ with the induced moment map Φ_i on the corresponding symplectic cut piece M_i. As a result, the moment image $\Phi(M)$ is the union of convex polytopes Δ_i.

Furthermore, if the circle fibers of the null fibration for a connected component $\mathcal{Z}$ of the fold Z are orbits for a circle subgroup $\mathbb{S}^1 \subset \mathbb{T}$, then $\Phi(\mathcal{Z})$ is a facet of each of the two polytopes corresponding to neighboring components of $M \setminus Z$. Let us denote these two polytopes Δ_1 and Δ_2. We note that they must **agree** near $\Phi(\mathcal{Z})$: there is a neighborhood $\mathcal{V}$ of $\Phi(\mathcal{Z})$ in $\mathbb{R}^n$ such that $\Delta_1 \cap \mathcal{V} = \Delta_2 \cap \mathcal{V}$. The condition that the circle fibers are orbits is automatically satisfied when the action is toric, and in that case there is a classification theorem in terms of the moment data.

The moment data of a toric origami manifold can be encoded in the form of an **origami template**. When the fold is coörientable, the template consists of a pair $(\mathcal{P}, \mathcal{F})$. The set $\mathcal{P}$ is a collection of Delzant polytopes corresponding to the Delzant polytopes of the symplectic cut pieces. The set $\mathcal{F}$ is a collection of pairs[5] of facets of polytopes in $\mathcal{P}$. These must satisfy

(1) if $\{F_1, F_2\}$ is in $\mathcal{F}$, with F_1 and F_2 facets of Δ_1 and Δ_2 respectively, then Δ_1 and Δ_2 agree near the facets F_1 and F_2; and
(2) if a facet occurs in a pair in $\mathcal{F}$, none of its neighboring facets or itself can occur in any other pair in $\mathcal{F}$.

The moment image of the origami manifold is the union of the polytopes in $\mathcal{P}$ under identification of the two facets in each pair in $\mathcal{F}$. The two identified facets are precisely the image of a connected component of the fold. We refer to these as **fold facets**. In the example of Figure 2.3, the collection $\mathcal{P}$ contains two identical isosceles right angle triangles, and $\mathcal{F}$ contains of only one pair, which consists of the hypotenuses of each of the triangles.

With these combinatorial data in place, we may now state the classification theorem.

Theorem 2.7 ([CGP, Theorem 3.13]). *There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{compact toric} \\ \textit{origami manifolds} \end{array} \right\} \leftrightsquigarrow \left\{ \textit{ origami templates } \right\},$$

up to equivariant origami symplectomorphism on the left-hand side, and affine equivalence on the right-hand side.

3. Cohomology concentrated in even degrees

Let M be a toric origami manifold with coörientable folding hypersurface and with origami template $(\mathcal{P}, \mathcal{F})$. Then $\mathcal{F}$ consists only of pairs of polytopes in $\mathcal{P}$. The **graph underlying the origami template** has vertices corresponding to polytopes in $\mathcal{P}$ and edges corresponding to the pairs in $\mathcal{F}$. We say that the origami template is **acyclic** if this graph is acyclic. A non-acyclic origami template is shown in Figure 3.1

We note that if M has an acyclic origami template, then M is automatically orientable. If the graph is acyclic, it is automatically a tree. The **leaves** of the origami template are those polytopes of $\mathcal{P}$ which are leaves in the graph.

The proof of the main theorem in this section will involve induction on the number of components in the origami template. To prove the inductive hypothesis, we need some auxiliary spaces. We focus on a connected component $\mathcal{Z}$ of the fold Z such that $M \smallsetminus \mathcal{Z}$ is the union of one open symplectic manifold W_- and one open origami manifold W_+. The corresponding closed manifolds

[5] When a component of the folding hypersurface is not coörientable, there is a corresponding singleton facet in $\mathcal{F}$.

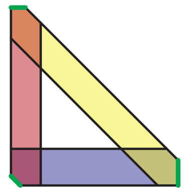

FIGURE 3.1. A non-acyclic template. Note that the corresponding manifold is not orientable, but the fold is coörientable.

with boundary are $M_- = W_- \cup \mathcal{Z}$ and $M_+ = W_+ \cup \mathcal{Z}$. Combinatorially, M_- corresponds to a leaf of the origami template for M.

Collapsing the fibers of the null-foliation on $\mathcal{Z}$ results in a toric symplectic manifold $\mathcal{B} = \mathcal{Z}/\mathbb{S}^1$ of dimension $\dim(\mathcal{B}) = \dim(M) - 2$. Cutting M along $\mathcal{Z}$ yields one toric symplectic manifold C_- and one toric origami manifold C_+ with one fewer connected component of the fold $Z \setminus \mathcal{Z}$. Finally, we use the space $C = C_+ \cup_{\mathcal{B}} C_-$, which is not a manifold. This notation is illustrated in the Figure 3.2 and summarized in Table 3.3.

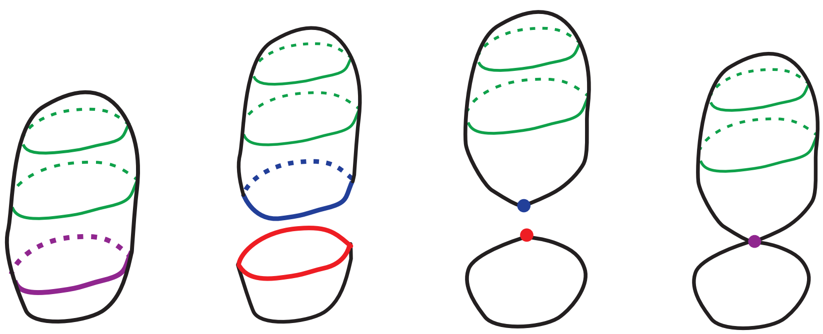

FIGURE 3.2. From left to right, the spaces M, $M_+ \sqcup M_-$, $C_+ \sqcup C_-$ and C.

TABLE 3.3. Summary of notation

Notation	Description
M	Toric origami manifold, $\mathbb{T} \circlearrowright M$
$\mathcal{Z} \subset Z$	Connected component $\mathcal{Z}$ of the fold Z
$\mathcal{B} \subset B$	Toric symplectic manifold $\mathcal{B} = \mathcal{Z}/\mathbb{S}^1$ and union of such $B = Z/\mathbb{S}^1$
W_+	Connected component of $M \setminus \mathcal{Z}$ that is an open origami manifold
M_+	$W_+ \cup \mathcal{Z}$, an origami manifold with boundary
C_+	$W_+ \cup \mathcal{B}$, an origami manifold with one less piece in its template
W_-	Connected component of $M \setminus \mathcal{Z}$ that is an open symplectic manifold
M_-	$W_- \cup \mathcal{Z}$, a symplectic manifold with boundary
C_-	$W_- \cup \mathcal{B}$, a toric symplectic manifold
C	$W_+ \cup \mathcal{B} \cup W_- = C_+ \cup_{\mathcal{B}} C_-$ (a $\mathbb{T}$-space, but not a manifold)

Lemma 3.4. *Suppose that* M *is a compact symplectic toric manifold with moment polytope* Δ_M*. Let* $\mathcal{B}$ *be a codimension* k *torus invariant symplectic submanifold whose moment map image* $\Delta_{\mathcal{B}}$ *is a* k*-dimensional face of* Δ_M*. Then the inclusion* $i : \mathcal{B} \hookrightarrow M$ *induces a surjection*

$$i^* : H^*(M;\mathbb{Z}) \twoheadrightarrow H^*(\mathcal{B};\mathbb{Z}).$$

Remark 3.5. Though it holds in more generality, we will only use this Lemma when the submanifold $\mathcal{B}$ is of codimension 2. Just as [MP, Lemma 2.3] allows Masuda and Panov to make inductive arguments, our Lemma 3.4 will be the crucial ingredient when we build the cohomology of M from its related toric pieces.

Proof. The manifold $\mathcal{B}$ is itself a symplectic toric manifold. Its cohomology is generated in degree 2, with one class for each facet F of $\Delta_{\mathcal{B}}$. Such a facet F is the intersection of a facet $\widetilde{F}$ of Δ_M with $\Delta_{\mathcal{B}}$. Under the restriction map i^*, the generator corresponding to $\widetilde{F}$ maps to the generator corresponding to F. Thus, i^* is surjective. □

Theorem 3.6. *Let* $\mathbb{T}\circlearrowright M$ *be a compact toric origami with acyclic origami template and coörientable folding hypersurface. Then the cohomology* $H^*(M;\mathbb{Z})$ *is concentrated in even degrees.*

Remark 3.7. Non-example 5.2 shows how the conclusion of the theorem might fail in the non-acyclic case.

Proof. We proceed by induction on the number n of connected components of the folding hypersurface. The base case is when $n = 0$, and M is a compact toric symplectic manifold. In this case, the fact that $H^*(M)$ is generated in degree 2, and hence concentrated in even degrees is well-known. For example, see [Da, J]. The case of a connected folding hypersurface is when $n = 1$, and is proven in [CGP, Corollary 5.1].

For the inductive step, we assume that every compact toric origami with acyclic origami template and coörientable folding hypersurface with at most $(n-1)$ connected components has cohomology concentrated in even degrees. Let M be a compact toric origami with acyclic origami template and coörientable folding hypersurface with n connected components.

Choose a leaf of the origami template, and let $\mathcal{Z}$ be the connected component of the folding hypersurface that corresponds to the facet separating the leaf from the rest of the origami template. We use the notation M_-, M_+, C_-, C_+, C and $\mathcal{B}$ as listed in Table 3.3. In particular, we note that C_- is actually a compact toric symplectic manifold and C_+ is a compact toric origami with acyclic origami template and coörientable folding hypersurface with $(n-1)$ connected components.

Let $\mathcal{Z} \xrightarrow{\pi} \mathcal{B}$ be the quotient by the null-fibration. Then π induces maps

$$M \xrightarrow{p} C \text{ and } M_- \xrightarrow{p_-} C_-.$$

We begin by studying the cohomology of C.

Claim 3.8. *The cohomology ring* $H^*(C;\mathbb{Z})$ *is concentrated in even degrees.*

Proof of Claim 3.8. We may choose $\mathbb{T}$-invariant collar neighborhoods of C_- and C_+ in C that deformation retract to C_- and C_+ respectively. This is analogous to choosing a collar neighborhood of Z in M, as described in the remarks just before Proposition 2.5 above.

The intersection of these neighborhoods is a collar neighborhood of $\mathcal{B}$ and deformation retracts onto $\mathcal{B}$. The Mayer-Vietoris sequence for these collar neighborhoods induces a long exact sequence, in cohomology with integer coefficients

$$\cdots \longrightarrow H^*(C) \longrightarrow H^*(C_+) \oplus H^*(C_-) \longrightarrow H^*(\mathcal{B}) \longrightarrow \cdots . \tag{3.9}$$

As C_- is a compact toric symplectic manifold, Lemma 3.4 implies that $H^*(C_-) \to H^*(B)$ is a surjection. Thus the long exact (3.9) splits into short exact sequences (again with integer coefficients)

$$0 \longrightarrow H^*(C) \longrightarrow H^*(C_+) \oplus H^*(C_-) \longrightarrow H^*(\mathcal{B}) \longrightarrow 0\,. \tag{3.10}$$

Note that the cohomology of C_- and $\mathcal{B}$ is concentrated in even degrees because C_- and $\mathcal{B}$ are compact toric symplectic manifolds. By the induction hypothesis, the cohomology of C_+ is concentrated in even degrees. We conclude from (3.10) in odd degrees that $H^*(C;\mathbb{Z})$ must be zero in odd degrees. ✔

We now look at the relationship between the cohomology of C_- and that of M_-.

Claim 3.11. *The quotient map* $p_- : M_- \longrightarrow C_-$ *induces a surjection in cohomology*

$$p_-^* : H^*(C_-;\mathbb{Z}) \twoheadrightarrow H^*(M_-;\mathbb{Z}).$$

In particular, $H^*(C_-;\mathbb{Z})$ *is concentrated in even degrees, and so* $H^*(M_-;\mathbb{Z})$ *is as well.*

Proof of Claim 3.11. This is an argument based on [HK, Proof of Proposition 1.3], with corrections following [H] and adjustments for integer coefficients. Consider long exact sequence in homology with integer coefficients of the pair $(C_-, \mathcal{B})$

$$\cdots \longrightarrow H_*(\mathcal{B}) \xrightarrow{i_*} H_*(C_-) \xrightarrow{j_*} H_*(C_-, \mathcal{B}) \longrightarrow \cdots\,, \tag{3.12}$$

where $i : \mathcal{B} \hookrightarrow C_-$ is inclusion and $j : (C_-, \emptyset) \longrightarrow (C_-, \mathcal{B})$ is inclusion of the pair. We may apply Poincaré duality to Lemma 3.4 to establish that i_* is an injection in homology with integer coefficients. Thus the long exact sequence (3.12) splits into short exact sequences. We then have a commutative diagram, with integer coefficients,

$$\begin{tikzcd}
 & H^{*-2}(\mathcal{B}) \arrow[r, "i_!"] \arrow[dd, "\cong"', "①"] & H^*(C_-) \arrow[r, "p_-^*"] \arrow[dd, "\cong"', "②"] & H^*(M_-) \arrow[d, "\cong"', "③"] & \\
 & & & H_{d-*}(M_-, \mathcal{Z}) \arrow[d, "\cong"', "④"] & \\
0 \arrow[r] & H_{d-*}(\mathcal{B}) \arrow[r, "i_*"] & H_{d-*}(C_-) \arrow[r, "j_*"] & H_{d-*}(C_-, \mathcal{B}) \arrow[r] & 0.
\end{tikzcd} \tag{3.13}$$

In this diagram, the manifold C_- has dimension d, and $\mathcal{B}$ has dimension $d-2$. The maps ① and ② are Poincaré duality for the manifolds $\mathcal{B}$ and C_- respectively, and ③ is Poincaré duality for the manifold M_- with boundary $\mathcal{Z}$. Finally, the map ④ is $(p_-)_*$ and is an isomorphism by excision. The left square commutes because it is the definition of the push-forward map $i_!$.

We now check that the right square commutes. We use the fact that the Poincaré duality isomorphism is the cap product with the fundamental class. So we need to show that for any $a \in H^*(C_-)$,

$$(p_-)_*\big(p_-^*(a) \frown [M_-]\big) = j_*\big(a \frown [C_-]\big).$$

But now, using the properties of the cap product as developed in [Ha, §3.3]

$$\begin{aligned}
(p_-)_*\big(p_-^*(a) \frown [M_-]\big) &= a \frown (p_-)_*\big([M_-]\big) && \text{by naturality of the cap product}\\
&= a \frown j_*\big([C_-]\big) && \text{because } (p_-)_*\big([M_-]\big) = j_*\big([C_-]\big)\\
&= j_*\big(a \frown [C_-]\big) && \text{by relative naturality of the cap product, and } j^*(a) = a.
\end{aligned}$$

Thus, we may now conclude that p_-^* is a surjection. ✔

Finally, we turn to the relationship between the cohomology of C and that of M.

Claim 3.14. *The quotient map* $p : M \longrightarrow C$ *induces a surjection in cohomology*

$$p^* : H^*(C;\mathbb{Z}) \twoheadrightarrow H^*(M;\mathbb{Z}).$$

Proof of Claim 3.14. We have long exact sequences in cohomology with integer coefficients for the pairs (M, M_-) and (C, C_-) that fit into a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(C, C_-) \arrow[r] \arrow[d, "\cong"', "①"] & H^*(C) \arrow[r] \arrow[d, "p^*"', "②"] & H^*(C_-) \arrow[r] \arrow[d, two heads, "p_-^*"', "③"] & H^{*+1}(C, C_-) \arrow[r] \arrow[d, "\cong"', "④"] & \cdots \\
\cdots \arrow[r] & H^*(M, M_-) \arrow[r] & H^*(M) \arrow[r] & H^*(M_-) \arrow[r] & H^{*+1}(M, M_-) \arrow[r] & \cdots .
\end{tikzcd} \tag{3.15}$$

Note that the maps ① and ④ are isomorphisms by excision, and the map ③ is onto by Claim 3.11. The Four Lemma states that if ① and ③ are onto and ④ is one-to-one, then ② must be onto. We have this for each degree, completing the proof. ✔

Claim 3.8 guarantees that the cohomology of C is concentrated in even degrees. Claim 3.14 tells us that $H^*(C;\mathbb{Z}) \xrightarrow{p^*} H^*(M;\mathbb{Z})$ is surjective, and so $H^*(M;\mathbb{Z})$ is necessarily concentrated in even degrees. □

4. Equivariant cohomology

Equivariant cohomology is a generalized cohomology theory in the equivariant category. We use the Borel model to compute equivariant cohomology. For the torus $\mathbb{T}$, we let $E\mathbb{T}$ be a contractible space on which $\mathbb{T}$ acts freely. Explicitly, for a circle, we may choose $E\mathbb{S}^1$ to be the unit sphere $\mathbb{S}^\infty$ in a Banach space. This is well-known to be contractible. Since $\mathbb{T} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a product, we may let $E\mathbb{T}$ be a product of infinite-dimensional spheres.

For any $\mathbb{T}$-space X, the diagonal action of $\mathbb{T}$ on $X \times E\mathbb{T}$ is free, and

$$X_{\mathbb{T}} = (X \times E\mathbb{T})/\mathbb{T}$$

is the **Borel mixing space** or **homotopy quotient** of X. We define the (Borel) equivariant cohomology ring to be

$$H^*_{\mathbb{T}}(X; R) := H^*(X_{\mathbb{T}}; R),$$

where $H^*(-; R)$ denotes singular cohomology with coefficients in the commutative ring R. Thus, when X is a free $\mathbb{T}$-space, we may identify

$$H^*_{\mathbb{T}}(X; R) \cong H^*(X/\mathbb{T}; R).$$

At the other extreme, if $\mathbb{T}$ acts trivially on X, then

$$H^*_{\mathbb{T}}(X; R) \cong H^*(X \times B\mathbb{T}; R),$$

where $B\mathbb{T} = E\mathbb{T}/\mathbb{T}$ is the **classifying space** of $\mathbb{T}$. Note that the cohomology of the classifying space, $H^*(B\mathbb{T}; R) \cong H^*_{\mathbb{T}}(pt; R)$, is the equivariant cohomology ring of a point.

For any $\mathbb{T}$-space X, we have the fibration

$$X \hookrightarrow X_{\mathbb{T}} \longrightarrow B\mathbb{T}. \tag{4.1}$$

The projection $X_{\mathbb{T}} \longrightarrow B\mathbb{T}$ induces the map $H^*_{\mathbb{T}}(pt; R) \longrightarrow H^*_{\mathbb{T}}(X; R)$, making $H^*_{\mathbb{T}}(X; R)$ an $H^*_{\mathbb{T}}(pt; R)$-module. Natural maps in equivariant cohomology preserve this module structure.

A common tool in the computation of equivariant cohomology is the Serre spectral sequence applied to the fibration (4.1). This has E_2-page

$$E_2^{p,q} = H^p(B\mathbb{T}; H^q(X; R)).$$

This spectral sequence converges to $H^*_{\mathbb{T}}(X; R)$. When X has cohomology concentrated in even degrees, then this spectral sequence is 0 in every other row and column, and automatically collapses. In particular, the equivariant cohomology is also concentrated in even degrees.

Goresky, Kottwitz and MacPherson call a $\mathbb{T}$-space X **equivariantly formal** if the Serre spectral sequence collapses at the E^2-page [GKM]. This spectral sequence does collapse for a compact toric origami manifold with acyclic origami template and coörientable folding hypersurface, because the cohomology is concentrated in even degrees (Theorem 3.6). Historically, the term **formal** has been used in rational homotopy theory, and so equivariantly formal has multiple interpretations. Scull describes the relationships between these interpretations [S]. To avoid further confusion, we will not use this term in the remainder of this paper.

Suppose that a torus $\mathbb{T}$ acts on a compact manifold M. Then the inclusion of the fixed points $I : M^{\mathbb{T}} \longrightarrow M$ induces a map in equivariant cohomology,

$$I^* : H^*_{\mathbb{T}}(M; R) \longrightarrow H^*_{\mathbb{T}}(M^{\mathbb{T}}; R). \tag{4.2}$$

A classical result of Borel establishes that the kernel and cokernel of I^* are torsion submodules [Br]. Our first step is to prove that in our set-up, I^* is injective. We can deduce this in a variety of ways. We supply a constructive proof here that we hope adds geometric intuition in the origami setting.

Theorem 4.3. *Let $\mathbb{T}\circlearrowright M$ be a compact toric origami with acyclic origami template and coörientable folding hypersurface. Then the inclusion $I : M^{\mathbb{T}} \hookrightarrow M$ induces an injection in equivariant cohomology*

$$I^* : H^*_{\mathbb{T}}(M; \mathbb{Z}) \longrightarrow H^*_{\mathbb{T}}(M^{\mathbb{T}}; \mathbb{Z}).$$

Proof. We proceed by induction on the number of components in the origami template.

Base Case: Suppose the template has a single component. Then M is a toric symplectic manifold. In particular, M is Kähler and has isolated fixed points. Frankel showed that $H^*(M; \mathbb{Z})$ is torsion free in this situation [Fr, Corollary 2]. The Serre spectral sequence then has no torsion at the E_2 page, where it collapses, so we may conclude that $H^*_{\mathbb{T}}(M; \mathbb{Z})$ is torsion free. As the fixed points are isolated, $H^*_{\mathbb{T}}(M^{\mathbb{T}}; \mathbb{Z})$ is also torsion free, and so Borel's classical result now implies injectivity.

Inductive Step: We now assume that the statement holds for any acyclic toric origami manifold with coörientable fold with at most $(n-1)$ polytopes in its origami template.

As in the previous section, we choose a leaf of the origami template, and let $\mathcal{Z}$ be the connected component of the folding hypersurface that corresponds to the facet separating the leaf from the rest of the origami template. We continue to use the auxiliary spaces M_-, M_+, C_-, C_+, C and $\mathcal{B}$ as listed in Table 3.3.

Claim 4.4. *The inclusion $C^{\mathbb{T}} \longrightarrow C$ induces an injection*

$$H^*_{\mathbb{T}}(C; \mathbb{Z}) \longrightarrow H^*_{\mathbb{T}}(C^{\mathbb{T}}; \mathbb{Z}).$$

Proof of Claim 4.4. We note that C_- is a toric symplectic manifold, and C_+ is a toric origami manifold with fewer pieces in its origami template. Thus, in equivariant cohomology with integer coefficients,

$$H^*_{\mathbb{T}}(C_-) \xrightarrow{I^*_-} H^*_{\mathbb{T}}(C^{\mathbb{T}}_-) \text{ and } H^*_{\mathbb{T}}(C_+) \xrightarrow{I^*_+} H^*_{\mathbb{T}}(C^{\mathbb{T}}_+)$$

are both injective.

We now consider the equivariant Mayer-Vietoris long exact sequence for $\mathbb{T}$-invariant neighborhoods of $C = C_+ \cup C_-$. The spaces C, C_+, C_- and $\mathcal{B}$ each have ordinary cohomology only in even degrees, and hence equivariant cohomology only in even degrees. Thus, the equivariant

Mayer-Vietoris long exact sequence splits into short exact sequences. We then have a commutative diagram, with integer coefficients,

$$
\begin{tikzcd}
0 \arrow[r] & H^*_{\mathbb{T}}(C) \arrow[r, "②"] \arrow[d, "①"'] & H^*_{\mathbb{T}}(C_+) \oplus H^*_{\mathbb{T}}(C_-) \arrow[r] \arrow[d, "③"'] & H^*_{\mathbb{T}}(\mathcal{B}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^*_{\mathbb{T}}(C^{\mathbb{T}}) \arrow[r, "④"'] & H^*_{\mathbb{T}}(C_+^{\mathbb{T}}) \oplus H^*_{\mathbb{T}}(C_-^{\mathbb{T}}) \arrow[r] & H^*_{\mathbb{T}}(\mathcal{B}^{\mathbb{T}}) \arrow[r] & 0
\end{tikzcd} \tag{4.5}
$$

The map ② is injective because the top row is short exact. The map ③ is $I_-^* \oplus I_+^*$, and is thus injective. Therefore, ③ $\circ$ ② is injective. But ③ $\circ$ ② $=$ ④ $\circ$ ①. Hence, ① must be injective. ✔

Claim 4.6. *In even degrees, the map*

$$H^{2*}_{\mathbb{T}}(C, C_-) \longrightarrow H^{2*}_{\mathbb{T}}(C^{\mathbb{T}}, C_-^{\mathbb{T}})$$

is injective.

Proof of Claim 4.6. The pair (C, C_-) is $\mathbb{T}$-invariant, so we consider the long exact sequence of the pair in equivariant cohomology. By Claim 3.8, the cohomology of C is concentrated in even degrees. The space C_- is a toric symplectic manifold, so its cohomology is also concentrated in even degrees. Thus the long exact sequence splits into a 4-term short exact sequence. This induces a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & H^{2*}_{\mathbb{T}}(C, C_-) \arrow[r, "②"] \arrow[d, "①"'] & H^{2*}_{\mathbb{T}}(C) \arrow[r] \arrow[d, "③"'] & H^{2*}_{\mathbb{T}}(C_-) \arrow[r] \arrow[d] & H^{2*+1}_{\mathbb{T}}(C, C_-) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{2*}_{\mathbb{T}}(C^{\mathbb{T}}, C_-^{\mathbb{T}}) \arrow[r, "④"] & H^{2*}_{\mathbb{T}}(C^{\mathbb{T}}) \arrow[r] & H^{2*}_{\mathbb{T}}(C_-^{\mathbb{T}}) \arrow[r] & H^{2*+1}_{\mathbb{T}}(C^{\mathbb{T}}, C_-^{\mathbb{T}}) \arrow[r] & 0
\end{tikzcd} \tag{4.7}
$$

The map ② is injective because the top row is exact. The map ③ is injective by Claim 4.4. Therefore, ③ $\circ$ ② is injective. But ③ $\circ$ ② $=$ ④ $\circ$ ①. Hence, ① must be injective. ✔

Claim 4.8. *The inclusion $M_-^{\mathbb{T}} \hookrightarrow M_-$ induces an injection $H^*_{\mathbb{T}}(M_-) \hookrightarrow H^*_{\mathbb{T}}(M_-^{\mathbb{T}})$.*

Proof of Claim 4.8. Recall that C_- is a toric symplectic manifold. Let $f : C_- \longrightarrow \mathbb{R}$ be the component of its moment map that attains its maximum value on $\mathcal{B}$. Let $f(\mathcal{B}) = b \in \mathbb{R}$. Let $g : M_- \longrightarrow \mathbb{R}$ be the composition $M_- \xrightarrow{p_-} C_- \xrightarrow{f} \mathbb{R}$. Choose $\varepsilon > 0$ such that there is no critical value in between $b - \varepsilon$ and b, and so that $g^{-1}((b - \varepsilon, b])$ is contained in the intersection of M_- with a Moser neighborhood of Z in M.

The fact that f is a Morse-Bott function on C_- with no critical values between $b - \varepsilon$ and b guarantees that $f^{-1}((-\infty, b))$ and $f^{-1}((-\infty, b-\frac{\varepsilon}{2}])$ are homotopy equivalent. The fact that $g^{-1}((b-\varepsilon, b])$ is contained in the intersection of M_- with a Moser neighborhood of $\mathcal{Z}$ in M guarantees that $f^{-1}((-\infty, b - \frac{\varepsilon}{2}])$ is homotopy equivalent to M_-.

We now appeal to a standard argument from equivariant symplectic geometry to conclude that

$$M_-^{\mathbb{T}} = f^{-1}\left(\left(-\infty, b - \frac{\varepsilon}{2}\right]\right)^{\mathbb{T}} \hookrightarrow f^{-1}\left(\left(-\infty, b - \frac{\varepsilon}{2}\right]\right) \simeq M_-$$

induces an injection in equivariant cohomology. This is an inductive argument on the critical set of f, and can be copied verbatim from the proof of [TW, Theorem 2]. ✔

We now consider the long exact sequence in equivariant cohomology for the pair (M, M_-). We have shown that M_- and M have cohomology concentrated in even degrees. Thus the long exact

sequence splits into a 4-term short exact sequence. This induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "①"] & H^{2*}_{\mathbb{T}}(M, M_-) \arrow[r] \arrow[d, "②"] & H^{2*}_{\mathbb{T}}(M) \arrow[r] \arrow[d, "③"] & H^{2*}_{\mathbb{T}}(M_-) \arrow[r] \arrow[d, "④"] & H^{2*+1}_{\mathbb{T}}(M, M_-) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^{2*}_{\mathbb{T}}(M^{\mathbb{T}}, M^{\mathbb{T}}_-) \arrow[r] & H^{2*}_{\mathbb{T}}(M^{\mathbb{T}}) \arrow[r] & H^{2*}_{\mathbb{T}}(M^{\mathbb{T}}_-) \arrow[r] & H^{2*+1}_{\mathbb{T}}(M, M_-) \arrow[r] & 0 \end{tikzcd}.$$

We first note that $H^*_{\mathbb{T}}(M, M_-) \cong H^*_{\mathbb{T}}(C, C_-)$, and $H^*_{\mathbb{T}}(M^{\mathbb{T}}, M^{\mathbb{T}}_-) = H^*_{\mathbb{T}}(C^{\mathbb{T}}, C^{\mathbb{T}}_-)$. Thus, the map ② is injective (in even degrees) by Claim 4.6. The map ④ is injective by Claim 4.8. The map ① is obviously surjective. Thus, by the Four Lemma, the map ③ must be injective, as desired. □

Remark 4.9. We may also derive Theorem 4.3 from work of Franz and Puppe [FP]. We describe this approach, and its further applications, in the proof of Theorem 4.14 below.

We now identify the image of I^*. Goresky, Kottwitz, and MacPherson proved that the equivariant cohomology of certain spaces may be described combinatorially as n-tuples of polynomials with divisibility conditions on pairs of the polynomials [GKM, Theorem 1.22]. The description applies, for example, to toric varieties, hypertoric varieties, and coadjoint orbits. In this section, we prove that the description also applies to any compact toric origami manifold with acyclic origami template and coörientable folding hypersurface. We begin by recalling the assumptions and results from [GKM]. The two simplifying assumptions in [GKM] are

(A) The fixed point set $M^{\mathbb{T}}$ consists of isolated points; and
(B) The **one-skeleton** $M_1 = \{p \in M \mid \dim(\mathbb{T} \cdot p) \leq 1\}$ is 2-dimensional.

The first assumption simplifies what $H^*_{\mathbb{T}}(M^{\mathbb{T}}; \mathbb{Z})$ can be. When the fixed point set consists of isolated points, this ring is a direct product of copies of

$$H^*_{\mathbb{T}}(pt; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n],$$

one for each fixed point. Thus, every class can be represented as a tuple of polynomials, and the ring structure is the component-wise product of polynomials.

When M is a compact Hamiltonian $\mathbb{T}$-space, the second assumption ensures that the one-skeleton must consist of 2-spheres intersecting one another at the isolated fixed points. Moreover, the $\mathbb{T}$-action preserves M_1, and the action rotates each $\mathbb{S}^2$ about an axis. The image of M_1 under the moment map is an immersed graph $\Phi(M_1) = \Gamma$ called the **moment graph**[6] whose vertices correspond to the fixed points $M^{\mathbb{T}}$ and whose edges correspond to the embedded $\mathbb{S}^2$'s. Each edge e in Γ is labeled by the weight [7] $\alpha_e \in \mathfrak{t}^*$ by which $\mathbb{T}$ acts on e. Indeed, the moment map sends the corresponding $\mathbb{S}^2$ to a line segment parallel to the weight α_e. The embedding of the graph Γ encodes, in this way, the **isotropy data**, denoted α. In this framework, we have the following description of $H^*_{\mathbb{T}}(M; \mathbb{Q})$.

Theorem 4.10 (Goresky-Kottwitz-MacPherson [GKM])**.** *Suppose M is a compact Hamiltonian $\mathbb{T}$-space satisfying conditions* (A) *and* (B) *above. Then I^* is injective*

$$I^* : H^*_{\mathbb{T}}(M; \mathbb{Q}) \hookrightarrow H^*_{\mathbb{T}}(M^{\mathbb{T}}; \mathbb{Q}) \cong \bigoplus_{p \in M^{\mathbb{T}}} H^*_{\mathbb{T}}(pt; \mathbb{Q}),$$

and its image consists of

$$\left\{ (f_p) \in \bigoplus_{p \in M^{\mathbb{T}}} H^*_{\mathbb{T}}(pt; \mathbb{Q}) \,\middle|\, \alpha_e \big| (f_p - f_q) \text{ for each edge } e = (p, q) \text{ in } \Gamma \right\}. \tag{4.11}$$

We will refer to these divisibility conditions as the **GKM description***.*

[6] The moment graph Γ is sometimes called the **GKM graph**. It is not the graph underlying the origami template.

[7] This is well-defined up to a sign, which is sufficient for our purposes.

Remark 4.12. For a Hamiltonian $\mathbb{T}$-space, assumption (A) guarantees that I^* is injective in equivariant cohomology with integer coefficients. We may strengthen assumption (B) to guarantee that the GKM description holds over $\mathbb{Z}$. A stronger set of assumptions are described in [HHH, §3]; they include the existence of a cell decomposition of the manifold. In particular, for Hamiltonian $\mathbb{T}$-spaces with isolated points, Morse theory can be applied to a generic component of the moment map to establish that these stronger assumptions boil down to local topological properties that must be checked at the fixed points. These can then be verified for symplectic toric manifolds and for coadjoint orbits.

As we have seen, the moment map for a toric origami manifold M does not necessarily produce Morse functions on M. We do not know if there is a cell decomposition of a toric origami manifold that would allow us to apply [HHH].

A key technical tool in the proof of Theorem 4.10 is the Chang-Skjelbred Lemma [CS, Lemma 2.3]. Let $J : M^{\mathbb{T}} \longrightarrow M_1$ denote the inclusion of the fixed points into the one-skeleton. The Chang-Skjelbred Lemma states that $I^*(H^*_{\mathbb{T}}(M)) = J^*(H^*_{\mathbb{T}}(M_1))$. Since the one-skeleton consists of $\mathbb{S}^2$'s, we must understand $H^*_{\mathbb{T}}(\mathbb{S}^2)$. It is a simple calculation to check that each $\mathbb{S}^2$ contributes one of the divisibility conditions in (4.11).

Now let $\mathbb{T} \circlearrowright M$ be a compact toric origami with acyclic origami template and coörientable folding hypersurface. The fixed points $M^{\mathbb{T}}$ correspond to the vertices of the origami template that do not appear on a fold facet. Just as for toric symplectic manifolds, these are isolated fixed points. The one-skeleton corresponds to the (possibly folded) edges (1-dimensional faces) of the origami template that are not entirely contained in a fold facet. The corresponding subsets of M are symplectic and origami 2-spheres. Therefore the one-skeleton is 2-dimensional. An example is shown in Figure 4.13.

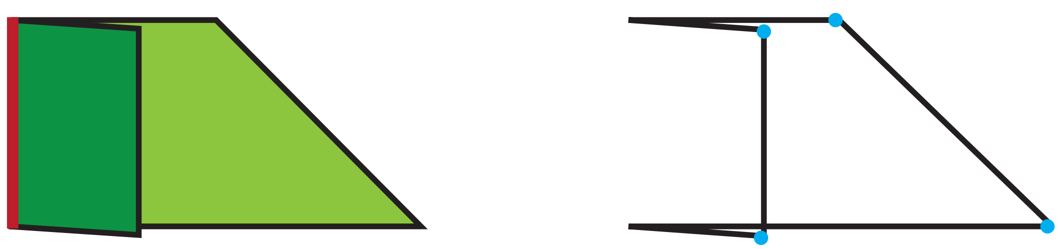

FIGURE 4.13. The origami template and GKM graph for a toric origami structure on the Hirzebruch surface. The GKM graph has four vertices and four edges, two of which are folded.

Thus, assumptions (A) and (B) are satisfied in the case of toric origami manifolds, and indeed the GKM theorem generalizes to our set-up.

Theorem 4.14. *Let $\mathbb{T} \circlearrowright M$ be a compact toric origami with acyclic origami template and coörientable folding hypersurface. Then I^* is injective*

$$I^* : H^*_{\mathbb{T}}(M;\mathbb{Z}) \hookrightarrow H^*_{\mathbb{T}}(M^{\mathbb{T}};\mathbb{Z}) \cong \bigoplus_{p\in M^{\mathbb{T}}} H^*_{\mathbb{T}}(pt;\mathbb{Z}),$$

and the image consists of

$$\left\{ (f_p) \in \bigoplus_{p\in M^{\mathbb{T}}} H^*_{\mathbb{T}}(pt;\mathbb{Z}) \;\middle|\; \alpha_e \big| (f_p - f_q) \text{ for each edge } e=(p,q) \text{ in } \Gamma \right\}, \tag{4.15}$$

where α_e is the weight of the action $\mathbb{T} \circlearrowright \mathbb{S}^2_e$ on the 2-sphere corresponding to e.

Proof. In Theorem 4.3, we have established that I^* is injective (over $\mathbb{Z}$). This can also be derived from an algebraic result of Franz and Puppe. In [FP, Theorem 1.1], for a $\mathbb{T}$-space X with connected stabilizers, they show that five conditions are equivalent. Their condition (ii) is that the Serre spectral sequence collapses at the E_2-page. Their condition (v) gives a long exact sequence.

The origami template classification of toric origami manifolds provides a proof that the stabilizer of a point is a connected subtorus of $\mathbb{T}$. Thus, we may appeal to Franz and Puppe's theorem. Our Theorem 3.6 implies that the Serre spectral sequence collapses at the E_2-page, assertion (ii) in [FP, Theorem 1.1]. This is then equivalent to assertion (v) which gives a long exact sequence, the first few terms of which are

$$0 \longrightarrow H^*_{\mathbb{T}}(M;\mathbb{Z}) \xrightarrow{①} H^*_{\mathbb{T}}(M_0;\mathbb{Z}) \xrightarrow{②} H^{*+1}_{\mathbb{T}}(M_1, M_0;\mathbb{Z}).$$

The content of our Theorem 4.3 is that ① (which is I^*) is injective. That the sequence is exact then means that the image of ① is equal to the kernel of ②. The map ② is the boundary map in the long exact sequence of the pair (M_1, M_0). Thus we have

$$\cdots \longrightarrow H^*_{\mathbb{T}}(M_1^*;\mathbb{Z}) \xrightarrow{③} H^*_{\mathbb{T}}(M_0;\mathbb{Z}) \xrightarrow{②} H^{*+1}_{\mathbb{T}}(M_1, M_0;\mathbb{Z}) \longrightarrow \cdots .$$

The kernel of ② is then equal to the image of ③, which is the image of the equivariant cohomology of the one-skeleton in $H^*_{\mathbb{T}}(M_0;\mathbb{Z})$. The fact that the one-skeleton consists of symplectic and origami 2-spheres means that each $\mathbb{S}^2$ contributes one of the divisibility conditions in (4.15). ☐

In Section 3, we proved that $H^*(M;\mathbb{Z})$ is concentrated in even degrees. We do not have a Morse function on M that would allow us to compute the ranks of these cohomology groups. With our explicit description of $H^*_{\mathbb{T}}(M;\mathbb{Z})$, it is possible in examples to determine the ranks and ring structure of $H^*(M;\mathbb{Z})$. This is a consequence of the collapse of the Serre spectral sequence, which implies that

$$H^*(M;\mathbb{Z}) \cong H^*_{\mathbb{T}}(M;\mathbb{Z}) \otimes_{H^*_{\mathbb{T}}(pt;\mathbb{Z})} \mathbb{Z}.$$

5. Example and non-example

Example 5.1. The $2n$-sphere $\mathbb{S}^{2n}$ may be endowed with toric origami structure whose origami template has two polytopes glued along a single facet. Each polytope is an n-simplex in $\mathbb{R}^n$ with an orthogonal corner at the origin, that is, a simplex with vertices the origin and the standard basis vectors $e_i = (0, \ldots, 1, \ldots, 0)$ with a single 1 in the i^{th} coördinate and 0s elsewhere. The fold facet is the $(n-1)$-simplex with vertices the e_i, opposite the origin. The origami template for $\mathbb{S}^4$ is shown in Figure 2.3.

Thus the toric action has 2 fixed points, which we denote N and S (for the north and south poles). There are n edges in the GKM graph, each joining $\Phi(N)$ and $\Phi(S)$. We can identify $H^*_{\mathbb{T}}(pt;\mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]$. From the origami template we can see that the $\mathbb{T}$ action on the sphere mapping to the i^{th} coördinate line in $\mathbb{R}^n$ has weight x_i. Theorem 4.14 states that

$$I^*(H^*_{\mathbb{T}}(\mathbb{S}^{2n};\mathbb{Z})) = \left\{ (f_N, f_S) \in \mathbb{Z}[x_1, \ldots, x_n] \oplus \mathbb{Z}[x_1, \ldots, x_n] \;\middle|\; x_i \middle| (f_N - f_S) \text{ for } i = 1, \ldots, n \right\}.$$

From this, we can find a module basis (for $H^*_{\mathbb{T}}(\mathbb{S}^{2n};\mathbb{Z})$ as an $H^*_{\mathbb{T}}(pt;\mathbb{Z})$-module) with two elements

$$I^*(\mathbb{1}) = (1, 1) \text{ and } I^*(\pi) = (x_1 \cdots x_n, 0),$$

where $\mathbb{1} \in H^0_{\mathbb{T}}(\mathbb{S}^{2n};\mathbb{Z})$ and $\pi \in H^{2n}_{\mathbb{T}}(\mathbb{S}^{2n};\mathbb{Z})$.

Non-example 5.2. The torus $\mathbb{T}^2$ is a toric origami manifold. The (toric) circle action is rotation along one of the coordinate circles. The folding hypersurface consists of two disjoint circles, as shown in Figure 5.3. The origami template consists of two intervals, glued to one another at each

end. The graph underlying this template has two vertices connected two one another by two edges, and therefore the template is not acyclic.

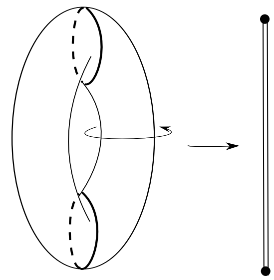

FIGURE 5.3. The moment map for $\mathbb{S}^1$ acting on $\mathbb{T}^2$.

It is not hard to compute

$$H^k(\mathbb{T}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0,2 \\ \mathbb{Z}\oplus\mathbb{Z} & k=1 \\ 0 & \text{else} \end{cases},$$

and so the conclusion of Theorem 3.6 fails. Moreover, the circle action is free, and so has no fixed points. Nevertheless, we may compute

$$H^k_{\mathbb{S}^1}(\mathbb{T}^2;\mathbb{Z}) = H^k(\mathbb{T}^2/\mathbb{S}^1;\mathbb{Z}) = H^k(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & k=0,1 \\ 0 & \text{else} \end{cases}.$$

In particular, the conclusion of Theorem 4.3 cannot hold.

6. TORIC ORIGAMI MANIFOLDS ARE LOCALLY STANDARD

Theorem 3.6 also follows from the work of Masuda and Panov [MP]. Their theory is more general and their proofs algebraic. We now show that toric origami manifolds fit into their framework.

A **torus manifold** is a $2n$-dimensional closed connected orientable smooth manifold M with an effective smooth action of an n-dimensional torus $\mathbb{T}^n$ with non-empty fixed set. A torus manifold M is said to be **locally standard** if every point in M has an invariant neighbourhood U weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ invariant under the standard $\mathbb{T}^n$-action on $\mathbb{C}^n$. The adverb 'weakly' means that there is an automorphism $\psi\colon \mathbb{T} \longrightarrow \mathbb{T}$ and a diffeomorphism $f\colon U \longrightarrow W$ such that

$$f(ty) = \psi(t)f(y)$$

for all $t \in \mathbb{T},\ y \in U$.

That compact symplectic toric manifolds are locally standard is widely accepted, but we could not find an explicit proof in the literature. We outline a proof here, using a local normal form theorem. Suppose $\mathbb{T} \circlearrowright M$ is a compact toric symplectic manifold, and let $p \in M$. Let $\mathbb{T}_0$ denote the stabilizer of p. Following [HNP, Lemma 2.1], there is a splitting $\mathbb{T} = \mathbb{T}_0 \times \mathbb{T}_1$, a symplectic vector space V, and a $\mathbb{T}$-equivariant symplectic open covering from an open set $\mathcal{U} \subset \mathbb{T}_1 \times \mathfrak{t}_1^* \times V$ to a neighborhood $\mathcal{V}$ of orbit $\mathbb{T}\cdot p$. The action $\mathbb{T} \circlearrowright \mathbb{T}_1 \times \mathfrak{t}_1^* \times V$ is given by

$$\begin{array}{rcl} (\mathbb{T}_0 \times \mathbb{T}_1) \times (\mathbb{T}_1 \times \mathfrak{t}_1^* \times V) & \longrightarrow & (T_1 \times \mathfrak{t}_1^* \times V) \\ ((t_0,t_1),(g,\gamma,\nu)) & \mapsto & (t_1\cdot g, \gamma, \rho(t_0)\nu), \end{array}$$

where $\rho : \mathbb{T}_0 \longrightarrow Sp(V)$ is a linear symplectic representation.

Now because M is a toric sympelctic manifold, the action $\mathbb{T} \circlearrowright M$ is effective, and so the open covering described above is actually a symplectomorphism. Moreover, $\dim M = 2{\cdot}\dim(\mathbb{T})$ implies

that $\dim(V) = 2 \cdot \dim(\mathbb{T}_0)$. The Delzant condition on the momentum polytope of M guarantees that there is an automorphism $\psi : \mathbb{T}_0 \longrightarrow \mathbb{T}_0$ so that the composition with $\rho : \mathbb{T}_0 \longrightarrow Sp(V)$ gives the standard action of $\mathbb{T}_0 \circlearrowright V \cong \mathbb{C}^{\dim(\mathbb{T}_0)}$. Finally, we may identify $\mathbb{T}_1 \times \mathfrak{t}_1^* \cong (\mathbb{C}^*)^{\dim(\mathbb{T}_1)} \subset \mathbb{C}^{\dim(\mathbb{T}_1)}$. Thus we conclude that the action $\mathbb{T} \circlearrowright M$ is indeed locally standard.

Next we prove that toric origami manifolds are locally standard in the coörientable setting. When the fold is not coörientable, it may be possible to prove the 'locally standard' condition by working in the oriented (and hence coörientable) double cover.

Lemma 6.1. *Suppose that* $(M, Z, \omega, \Phi, \mathbb{T})$ *is a toric origami manifold such that each connected component of the folding hypersurface is coörientable. Then* M *is locally standard.*

Proof. The 'locally standard' condition must be checked locally. We first consider points not on the fold Z. We may perform symplectic cuts to M resulting in its symplectic cut pieces $M_1, \dots, M_k$, each of which is a compact toric symplectic manifold. Thus, each point p in $M \setminus Z$ has a neighborhood $U_p \subset M \setminus Z$ that is equivariantly diffeomorphic to an open set $\widetilde{U}_p$ in one of the symplectic cut pieces M_i. As outlined above, compact symplectic toric manifolds are locally standard, so by possibly passing to a smaller neighborhood, we have a neighborhood of p that is weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ that is invariant with respect to the standard $\mathbb{T}^n$-action on $\mathbb{C}^n$.

Now we turn to a point $p \in Z$ on the fold. We use a Moser model, as defined in [CGP, Def. 2.12], for a neighborhood of p. As remarked in [CGP], such Moser models exist for orientable origami manifolds. What is necessary for the local existence of the Moser model near a single component of the fold is simply the coörientability of that piece of the fold. Thus, we may assume that $p \in Z$ has a neighborhood with a Moser model.

Let Z_p denote the connected component of Z containing p. The local Moser model is an equivariant diffeomorphism

$$\varphi : Z_p \times (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U},$$

where $\varepsilon > 0$ and $\mathcal{U}$ is a tubular neighborhood of Z_p, such that $\varphi(x, 0) = x$ for all $x \in Z_p$. The symplectic form can be written in these coördinates, but we do not need that here.

We now consider the null-fibration $\mathbb{S}^1 \hookrightarrow Z_p \xrightarrow{\pi} B_p$. This is a principal $\mathbb{S}^1$-bundle, and the base space is a compact symplectic toric manifold of dimension $(2n-2)$. Let $b = \pi(p)$. Compact toric symplectic manifolds are locally standard. Choose a neighborhood V of $b \in B_p$ that is weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^{n-1}$ that is invariant with respect to the standard $\mathbb{T}^{n-1}$-action on $\mathbb{C}^{n-1}$. By possibly passing to a smaller neighborhood of b, we may assume that the bundle over V is trivial, $V \times \mathbb{S}^1 \xrightarrow{\pi} V$. Thus, we have an equivariant neighborhood

$$V \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$$

of $p \in Z_p$. Under this identification, the action of $\mathbb{T}^n$ splits into the $\mathbb{T}^{n-1}$ action on V, and $\mathbb{S}^1$ acting on itself by multiplication on the $\mathbb{S}^1$. We may embed $\mathbb{S}^1 \times (-\varepsilon, \varepsilon)$ as an open annulus $A \subset \mathbb{C}$ by equivariant diffeomorphism. Therefore $V \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$ is weakly equivariantly diffeomorphic to an open subset $W \times A \subset \mathbb{C}^{n-1} \times \mathbb{C}$ that is invariant with respect to the coördinate $\mathbb{T}^n$-action on the vector space $\mathbb{C}^n$. □

A key player in Masuda and Panov's work on torus manifolds is the orbit space $Q = M/\mathbb{T}$. In the origami framework, this orbit space is closely related to the origami template $(\mathcal{P}, \mathcal{F})$. Indeed, when $\mathbb{T} \circlearrowright M$ is a toric origami manifold, the orbit space is the quotient

$$M/\mathbb{T} = \bigsqcup_{\Delta \in \mathcal{P}} \Delta \Big/ F_1 \sim F_2 \quad \text{for all pairs } \{F_1, F_2\} \in \mathcal{F}.$$

Masuda and Panov define the **faces** of the orbit space using their notion of **characteristic submanifold**. The orbit space is then called **face-acyclic** if every face F (including Q itself) is **acyclic**: that is, it has $\widetilde{H}^*(F) = 0$.

Unravelling their definitions in our context, we have the following combinatorial description of the faces of $M/\mathbb{T}$ in terms of the origami template. The facets of a polytope are well-understood. The facets of $M/\mathbb{T}$ may be described in terms of the facets in $\mathscr{P} = \coprod_{\Delta \in \mathcal{P}} \Delta$. Since $\mathscr{P}$ is a disjoint union of polytopes, its facets are the collection of facets of each polytope $\Delta \in \mathcal{P}$. In relation to $\mathcal{F}$, and following the rules governing an origami template, the facets of $\mathscr{P}$ come in three different flavors:

(1) $\mathscr{F}_1$ the set of facets that appear in a pair in $\mathcal{F}$;
(2) $\mathscr{F}_2$ the set of facets that neighbor a facet in a pair in $\mathcal{F}$; and
(3) $\mathscr{F}_3$ the set of facets that do not neighbor any facet in a pair in $\mathcal{F}$.

That is, the set of facets of $\mathscr{P}$ is a disjoint union

$$\mathscr{F}_1 \sqcup \mathscr{F}_2 \sqcup \mathscr{F}_3.$$

The set of **facets** of $M/\mathbb{T}$ is then

$$\left(\mathscr{F}_2/\sim\right) \sqcup \mathscr{F}_3,$$

where the identification $\sim$ is induced by identifying the pairs in $\mathcal{F}$. The **faces** of $M/\mathbb{T}$ are then non-empty intersections of facets in $M/\mathbb{T}$, together with $M/\mathbb{T}$ itself.

Theorem 6.2. *Suppose that $(M, Z, \omega, \Phi, \mathbb{T})$ is a toric origami manifold such that each connected component of the folding hypersurface is coörientable. Then $M/\mathbb{T}$ is face-acyclic if and only if the origami template is acyclic.*

Proof. If $M/\mathbb{T}$ is face-acyclic, then it is acyclic itself, which can only happen if the graph underlying the origami template has no cycles.

Now, suppose $M/\mathbb{T}$ is not face-acyclic, and let F be a face of $M/\mathbb{T}$ which fails to be acyclic. If F is $M/\mathbb{T}$ itself, then the underlying graph must not be acyclic. If F is a proper face, it fails to be acyclic if and only if the collection of polytopes $\Delta_i \in \mathcal{P}$ that have non-empty intersection with F create a cycle in the graph underlying the origami template. This completes the proof. □

We now can derive our Theorem 3.6 from Masuda and Panov's work: they prove that face-acyclic locally standard torus manifolds have no odd-degree cohomology [MP, Theorem 9.3]. While our proofs have very different flavors, it is interesting to note that a crucial ingredient in their proof is their [MP, Lemma 2.3], which is closely related to our Lemma 3.4, as described in Remark 3.5.

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